DIFFRACTION OF A PLANE WAVE BY A SMOOTH RIGID WEDGE IN AN UNBOUNDED ELASTIC MEDIUM IN ABSENCE OF FRICTION

(DIFRAKTSIIA PLOSKOI VOLNY NA ZHESTKOM KLINE VSTAVLENNOM BEZ TRENIIA V BEZGRANICHNUIU UPRUGUIU SREDU)

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B.V. KOSTROV
(Moscow)

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Problems of diffraction of elastic waves have in recent times attracted more and more attention, particularly those related to the interaction of elastic waves with movable obstacles. If the obstacle has a polygonal form, the problem reduces to the investigation of the diffraction of an incident wave by an infinite wedge. While the wedge problem has been studied exhaustively in the case of diffraction of acoustic waves [1, 2 and 3], the corresponding problem in the theory of elasticity has, in general, not yet been solved analytically if we disregard the special case when the wedge degenerates into a semi-infinite cut [4].

In the present article another case is investigated, which permits the construction of an analytical solution when friction is absent between a rigid wedge and the surrounding elastic medium, but the medium nevertheless does not become detached from the wedge, i.e. when the normal displacement and the shearing stress disappear on the faces of the wedge. Under these conditions the reflection of an incident wave from the face of the wedge does not lead to the appearance of waves of different type. That is, in the case of an incident longitudinal wave only a longitudinal wave is reflected, and for an incident transverse wave only a transverse wave is reflected. However, as is known, if the boundary has a sharp edge the boundary and initial conditions are by themselves insufficient to guarantee uniqueness of solution. An additional condition must be formulated, a so-called 'edge condition', which is equivalent to the requirement that the law of conservation of energy be satisfied. At it turns out, this last condition cannot be satisfied if the disturbances are limited to a single type (longitudinal or transverse). This leads to the occurrence of waves of both types diffracted from the edge, despite the absence of two types of reflected waves. This last circumstance was not noticed in the paper by Sveklo and Siukiiainen [5] which was devoted to the problem considered here. The results obtained in that paper are, therefore, incorrect.

In spite of the fact that the edge condition does not permit complete reduction to the acoustic case, the problem nevertheless turns out to be very similar to the acoustic one, which makes it possible to find a closed analytical solution. The similarity of the problem to the acoustic one is, in the end, reflected in the form of the solution for the disturbing wave, which is a sum of two solutions. The first of these is simply the solution of the corresponding acoustic problem, while the second describes the effect of elasticity. As the investigation shows, the latter part of the solution can in no way be neglected in comparison to the acoustic part. Only in one special case, when a plane longitudinal wave is incident on the wedge along its bisector, the elastic term is absent and the solution coincides with the acoustic one.

1. Formulation of the problem. We shall consider an elastic medium with shear modulus μ and velocities of propagation of longitudinal and transverse waves a and b, repectively. The medium occupies the sector $r \geqslant 0,\quad 0 \leqslant \vartheta \leqslant \pi / k$ and is in contact with a rigid wedge $(\pi / k \leqslant \vartheta \leqslant 2\pi)$, the boundary condition having the form

$$u_\vartheta = 0, \qquad \tau_{r\vartheta} = 0 \qquad \text{for} \quad \vartheta = 0,\ \pi / k, \quad 0 \leqslant r \leqslant \infty \tag{1.1}$$

where r and θ are cylindrical coordinates. Without loss of generality it is possible to consider $k < 1$, since the case $k > 1$ can be obtained from the former one with the aid of a reflection transformation. For if we take the solution for $k < 1$ and separate out the part which is antisymmetric with respect to the bisector of the wedge, this latter part is the solution for $k' = 2k \geqslant 1$.

If we introduce longitudinal and transverse potentials which are related to the displacement components by the relations

$$u_r = \frac{\partial \varphi}{\partial r} + \frac{1}{r}\frac{\partial \psi}{\partial \vartheta}, \qquad u_\vartheta = \frac{1}{r}\frac{\partial \varphi}{\partial \vartheta} - \frac{\partial \psi}{\partial r} \tag{1.2}$$

then the boundary conditions (1.1) will be satisfied if we require that

$$\partial\varphi / \partial\vartheta = 0, \qquad \psi = 0 \qquad \text{for} \quad \vartheta = 0, \quad \pi / k \tag{1.3}$$

The boundary conditions are thus set up independently for the longitudinal and transverse potentials. This allows us to find the potentials independently until the edge condition is taken into account.

As the edge condition, we require that for $k < 1$ the displacements be bounded and the stresses and strains grow more slowly than r^{-1}, or in other words,

$$\mathbf{u} = O(r^\lambda) + \text{const}, \qquad \lambda > 0 \qquad \text{as} \quad r \to 0 \tag{1.4}$$

We shall consider that the potential of the incident wave is described by the Heaviside step function $H(\tau)$, i.e. it is equal to zero before the front and to unity behind it. Use of the Duhamel integral leads us to the case of a general plane incident wave. We shall split the unknown longitudinal and transverse potentials into two terms. The first of these will describe the incident wave. The second will represent the disturbance caused by the presence of the wedge and will contain the reflected and diffracted waves. We thus write

for a longitudinal incident wave

$$\varphi = H\left(\frac{at}{r} + \cos(\vartheta - \vartheta_0)\right) + \varphi_1, \qquad \psi = \psi_1 \tag{1.5}$$

for a transverse incident wave

$$\varphi = \varphi_1, \qquad \psi = H\left(\frac{bt}{r} + \cos(\vartheta - \vartheta_0)\right) + \psi_1 \tag{1.6}$$

The potentials φ_1 and ψ_1 describing the disturbance must, of couse, satisfy homogeneous initial conditions and the boundary conditions which follow from Eq. (1.3), i.e. the conditions for an incident longitudinal wave

$$\frac{\partial\varphi_1}{\partial\vartheta} = \sin(\vartheta - \vartheta_0)\,\delta\left(\frac{at}{r} + \cos(\vartheta - \vartheta_0)\right), \qquad \frac{\partial\psi_1}{\partial r} = 0 \quad \text{for} \quad \vartheta = 0, \quad \pi / k \tag{1.7}$$

for a transverse incident wave

$$\frac{\partial\varphi_1}{\partial\vartheta} = 0, \qquad \frac{\partial\psi_1}{\partial r} = -\frac{\cos(\vartheta - \vartheta_0)}{r}\,\delta\left(\frac{bt}{r} + \cos(\vartheta - \vartheta_0)\right) \tag{1.8}$$

where $\delta(\tau)$ is the Dirac delta function.

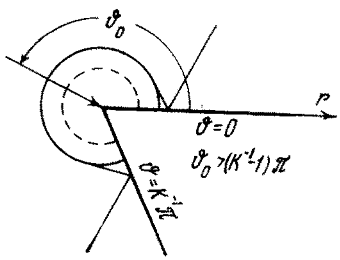

FIG. 1

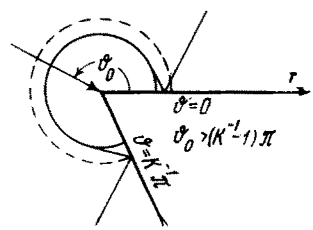

FIG. 2

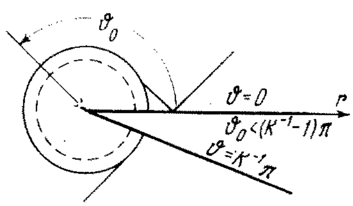

FIG. 3

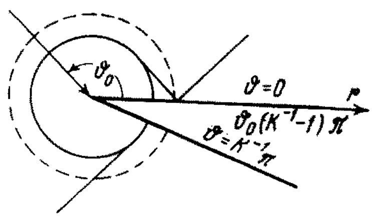

FIG. 4

The reflected and diffracted wave fronts are shown in Fig. 1 for the case of a longitudinal incident wave and in Fig. 2 for the case of a transverse incident wave, under the conditions that no shadow zone is formed. The cases where shadows are formed are shown in Figs. 3 and 4. We note that the sign of the potential of the reflected longitudinal wave coincides with the sign of the potential of the incident longitudinal wave (i.e. the reflection coefficient is equal to unity.) However, the potential of the reflected transverse wave has a sign opposite to that of the incident transverse wave (the reflection coefficient is equal to −1.)

The boundary and initial conditions are such that the solution which is sought must be homogeneous and of degree zero in r and t. This allows us to make use of the method of functionally invariant solutions of Smirnov and Sobolev [1]. Pursuing this method, we introduce the variables

$$z_1 = \cosh k(\chi_1 + i\vartheta), \qquad z_2 = \cosh k(\chi_2 + i\vartheta) \qquad \left(\cosh \chi_1 = \frac{at}{r}, \cosh_2 \chi_2 = \frac{bt}{r}\right) \tag{1.9}$$

The region of the longitudinal disturbance is mapped into the upper half-plane of the complex variable z_1 in the following way: the sector $0 \leqslant \vartheta \leqslant \pi / k$, $0 < r < at$ is mapped into the upper half-plane, the radii $\theta = 0$ and $\theta = \pi/k$ being transformed into the segments $(1, \infty)$ and $(-\infty, -1)$ of the real axis. The arc of the circle $r = at$ (the front of the diffracted longitudinal wave), is transformed into the segment $(-1, 1)$ of the real axis. The exterior of this sector (that part of it where longitudinal disturbances are possible) is also mapped onto the segment $(-1, 1)$. The region of transverse disturbances is mapped onto the upper half-plane of the variable z_2 in exactly the same way.

The potentials φ_1 and ψ_1 can now be sought in the form $\varphi_1 = \mathrm{Re}\, \Phi(z_1)$, $\psi_1 = \mathrm{Re}\, \Psi(z_2)$- where $\Phi(z_1)$ and $\Psi(z_2)$ are analytic functions of z_1 and z_2 which are regular in the upper half-plane and satisfy on the real axis conditions which follow from the boundary conditions (1.7) or (1.8).

2. A longitudinal incident wave. From (1.7) it can be shown that

$$\mathrm{Re}\,[i\sqrt{z_1^2 - 1}\,\Phi'(z_1)] = 0, \qquad \mathrm{Im}\, z_1 = +0 \tag{2.1}$$

and that at the points $z_1 = \cos k(\vartheta_0 - \pi)$ and $z_1 = \cos k(\vartheta_0 + \pi)$ the function $\Phi'(z_1)$ must have simple poles with residues equal to two in absolute value. The sign of the residue depends on the angle of incidence and, as can be verified, must agree with the sign of $\sin k(\vartheta_0 - \pi)$ at the pole $z_1 = \cos k(\vartheta_0 - \pi)$ and be opposite to the sign of $\sin k(\vartheta_0 + \pi)$ at the pole $z_1 = \cos k(\vartheta_0 + \pi)$. We choose the branch of the radical $(z_1^2 - 1)^{1/2}$ which is equal to $+i$ for $z_1 = 0$. The solution of the boundary value problem (2.1) can then be written in the form

$$\begin{aligned}\Phi'(z_1) = \frac{1}{\pi\sqrt{z_1^2 - 1}}\,[A_1(z_1)\sin(\vartheta_0 - \pi)k\,(z_1 - \cos(\vartheta_0 - \pi)k)^{-1} - \\ - A_2(z_1)\sin(\vartheta_0 + \pi)k\,(z_1 - \cos(\vartheta_0 + \pi)k)^{-1}]\end{aligned} \tag{2.2}$$

where A_1 and A_2 are polynomials of degree n_1 which satisfy the conditions

$$A_1(\cos(\vartheta_0 - \pi)k) = A_2(\cos(\vartheta_0 + \pi)k) = 1 \tag{2.3}$$

Taking into consideration that $z_1 = O(r^{-k})$ as $r \to 0$, we may conclude from the above that the displacement components are of the order

$$r^{[(1-n_1)k]-1} \qquad \text{as} \quad r \to 0$$

That is, they are not bounded, even when $n = 0$, and the edge condition (1.4) cannot be satisfied by a single longitudinal potential.

In order to satisfy the edge condition (1.4) we shall try to find a transverse potential Ψ which will cancel the displacement singularity corresponding to the longitudinal potential (2.2). It follows from Eqs. (1.7) that

$$\mathrm{Re}\,\Psi'(z_2) = 0 \qquad \text{for} \quad \mathrm{Im}\, z_2 = 0 \tag{2.4}$$

where $\Psi'(z_2)$ is regular throughout the upper half-plane, including the real axis. From this it follows that

$$\Psi'(z_2) = iB(z_2) \tag{2.5}$$

where B is a polynomial of degree n_2 having real coefficients. The corresponding displacement are of the order

$$r^{-[(n_2+1)k]-1} \quad \text{as} \quad r \to 0$$

In order that the singularity at the edge be annulled, we must have $n_1 - 1 = n_2 + 1$ or $n_2 = n_1 - 2$. It can be verified that the edge condition can be satisfied only for $n_2 = 0$ and $n_1 = 2$. Equations (2.2) and (2.5) can now be presented in the form

$$\begin{gathered}\Phi'(z_1) = \frac{1}{\pi\sqrt{z_1^2 - 1}}[(z_1 - \cos(\vartheta_0 - \pi)k)^{-1}\sin(\vartheta_0 - \pi)k - \\ - (z_1 - \cos(\vartheta_0 + \pi)k)^{-1}\sin(\vartheta_0 + \pi)k + \alpha z_1 + \beta], \qquad \Psi'(z_1) = \frac{i}{\pi}\gamma\end{gathered} \tag{2.6}$$

We find the expressions for the displacements from (1.2) and (2.6)

$$\begin{gathered}u_\vartheta = \frac{-k}{\pi r}\,\mathrm{Im}\left[(z_1 - \cos k(\vartheta_0 - \pi))^{-1}\sin k(\vartheta_0 - \pi) - \right. \\ \left. - (z_1 - \cos k(\vartheta_0 + \pi))^{-1}\sin k(\vartheta_0 + \pi) + \alpha z_1 + \beta + \frac{bt\gamma\sqrt{z_2^2 - 1}}{\sqrt{b^2t^2 - r^2}}\right] \\ u_r = \frac{-k}{\pi r}\,\mathrm{Re}\left\{\frac{at}{\sqrt{a^2t^2 - r^2}}[(z_1 - \cos k(\vartheta_0 - \pi))^{-1}\sin k(\vartheta_0 - \pi) - \right. \\ \left. - (z_1 - \cos k(\vartheta_0 + \pi))^{-1}\sin k(\vartheta_0 + \pi + \alpha z_1 + \beta)] + \gamma\sqrt{z_2^2 - 1}\right\}\end{gathered} \tag{2.7}$$

The real constants α, β, and γ in Eqs. (2.6) and (2.7) are determined from the edge condition. Using Eqs. (1.9) it is easy to find the leading terms of the asymptotic expansions of Eqs. (2.7) as $r \to 0$

$$\begin{gathered}u_\vartheta = \frac{k}{\pi r}\sin k\vartheta\left\{-a^k\frac{\alpha}{2}\left(\frac{2t}{r}\right)^k + b^k\frac{\gamma}{2}\left(\frac{2t}{r}\right)^k + \right. \\ \left. + a^{-k}\left[2\sin k(\vartheta_0 - \pi) - 2\sin k(\vartheta_0 + \pi) + \frac{\alpha}{2}\right]\left(\frac{r}{2t}\right)^k - b^{-k}\frac{\gamma}{2}\left(\frac{r}{2t}\right)^{-k}\right\} + o(1) \\ u_r = \frac{-k}{\pi r}\left\{\cos k\vartheta\left[a^k\frac{\alpha}{2}\left(\frac{2t}{r}\right)^k - b^k\frac{\gamma}{2}\left(\frac{2t}{r}\right)^k + \right.\right. \\ \left.\left. + a^{-k}\left(2\sin k(\vartheta_0 - \pi) - 2\sin k(\vartheta_0 + \pi) + \frac{\alpha}{2}\right)\left(\frac{r}{2t}\right)^k - b^{-k}\frac{\gamma}{2}\left(\frac{r}{2t}\right)^k\right] + \beta\right\} + o(1)\end{gathered}$$

It is clear from this that it is necessary to put

$$\beta = 0, \quad a^k\alpha + b^k\gamma = 0, \quad a^{-k}[4\sin k(\vartheta_0 - \pi) - 4\sin k(\vartheta_0 + \pi)] + a^{-k}\alpha - b^{-k}\gamma = 0$$

in order to satisfy the edge condition.

Now instead of (2.6) we obtain

$$\begin{gathered}\Phi'(z_1) = \frac{1}{\pi\sqrt{z_1^2 - 1}}\left\{(z_1 - \cos k(\vartheta_0 - \pi))^{-1}\sin k(\vartheta_0 - \pi) - \right. \\ \left. - (z_1 - \cos k(\vartheta_0 + \pi))^{-1}\sin k(\vartheta_0 + \pi) + 8\left[1 + \left(\frac{a}{b}\right)^{2k}\right]^{-1}\sin\pi k\cos k\vartheta_0 z_1\right\} \\ \Psi'(z_2) = -8i\left[\left(\frac{a}{b}\right)^k + \left(\frac{b}{a}\right)^k\right]^{-1}\sin\pi k\cos\vartheta_0 k\end{gathered} \tag{2.8}$$

Integrating and changing over to the physical variables, we obtain

$$\varphi = \varphi_a(r, \vartheta, t) + \frac{8}{\pi}\left[1 + \left(\frac{a}{b}\right)^{2k}\right]^{-1} \sin \pi k \cos \vartheta_0 k \cos \vartheta_0 k \left[P(at/r) - \frac{1}{P(at/r)}\right]$$

$$\psi = \frac{8}{\pi}\left(\frac{b}{a}\right)^{k}\left[1 + \left(\frac{a}{b}\right)^{2k}\right]^{-1} \sin \pi k \cos \vartheta_0 k \sin \vartheta k \left[P(bt/r) - \frac{1}{P(bt/r)}\right] \qquad (2.9)$$

where φ_a is the solution of the acoustic problem with the boundary condition $\partial\varphi_a / \partial\vartheta = 0$, and the function $P(\tau)$ is equal to

$$P(\tau) = (\tau + \sqrt{\tau^2 - 1})^k \qquad (2.10)$$

It is clear from these equations, that the corrections to the acoustic solution disappear on the fronts of the diffracted waves, which is quite natural since they are not related to the waves reflected from the faces of the wedge. It is also easy to see that these corrections satisfy homogeneous initial and boundary conditions. The solution obtained coincides with the acoustic one if the incident ray is directed along the bisector of the wedge ($\vartheta_0 = \pi / (2k)$), since in this case $\cos k\vartheta_0 = 0$, and both corrections vanish identically. For this value of the angle of incidence the acoustic solution satisfies the edge condition (1.4) by itself.

3. A transverse incident wave. If a transverse wave is incident, the conditions (2.1) and (2.4) remain valid, but now $\Phi'(z_1)$ must be regular in the half-plane including the real axis and $\Psi'(z_2)$ must have simple poles at the points $z_2 = \cos k(\vartheta_0 - \pi)$ and $z_2 = \cos k(\vartheta_0 + \pi)$ with residues equal to ± 2 respectively. Proceeding in a manner analogous to that described in the previous section, we find

$$\Phi'(z_1) = \frac{-8z_1}{\pi\sqrt{z_1^2 - 1}}\left[\left(\frac{a}{b}\right)^k + \left(\frac{b}{2}\right)^k\right]^{-1} \sin \pi k \sin \vartheta_0 k$$

$$\Psi'(z_2) = \frac{1}{\pi i}\Big\{[z_2 - \cos k(\vartheta_0 - \pi)]^{-1} - [z_2 - \cos k(\vartheta_0 + \pi)]^{-1} + 8\left[1 + \left(\frac{b}{a}\right)^{2k}\right]^{-1} \sin \pi k \sin \vartheta_0 k\Big\} \qquad (3.1)$$

We obtain from this the expressions for the potentials in the physical variables

$$\varphi = \frac{8}{\pi}\left[\left(\frac{a}{b}\right)^k \left(\frac{b}{a}\right)^k\right]^{-1} \sin \pi k \sin \vartheta_0 k \cos \vartheta k \left[P(at/r) - \frac{1}{P(at/r)}\right]$$

$$\psi = \psi_a + \frac{8}{\pi}\left[1 + \left(\frac{b}{a}\right)^{2k}\right]^{-1} \sin \pi k \sin \vartheta_0 k \sin \vartheta k \left[P(bt/r) - \frac{1}{P(bt/r)}\right] \qquad (3.2)$$

where ψ_a is again the solution of the corresponding acoustic problem ($\psi_a = 0$ at the faces of the wedge). In the present case there exists no value of the angle of incidence for which the solution coincides with the acoustic one, since $\sin k\theta_0$ vanishes only for $\theta_0 = 0$ or π/k, i.e. when the incident ray grazes one face of the wedge. However, this is not possible, since in this case the reflected wave cancels the incident wave.

4. Behavior of the diffracted waves near their fronts. By the use of Equations (1.2), (1.9), (1.10) and (2.8) with (3.1) it is easy to compute asymptotic expressions for the displacements near the fronts of the diffracted waves (but not near points of contact of these fronts with the fronts of the reflected or incident waves.)

We introduce the variables

$$\tau_1 = \frac{at}{r} - 1, \qquad \tau_2 = \frac{bt}{r} - 1 \tag{4.1}$$

which represent the distances from the longitudinal and transverse wave fronts, respectively. Then for a longitudinal incident wave, we have

for $\tau_1 \to 0$

$$u_\vartheta = O(1), \qquad u_r \approx \frac{-k}{\pi r \sqrt{2\tau_1}} \Big\{[\cos k\vartheta - \cos k(\vartheta_0 - \pi)]^{-1} \sin k(\vartheta_0 - \pi) - \tag{4.2}$$
$$- [\cos \vartheta k - \cos k(\vartheta_0 + \pi)]^{-1} \sin k(\vartheta_0 + \pi) + 8\left[1 + \left(\frac{a}{b}\right)^k\right]^{-1} \sin k\pi \cos k\vartheta_0 \cos \vartheta k\Big\} + O(1)$$

for $\tau_2 \to 0$ (i.e. near the front of the transverse wave)

$$u_r = O(1), \qquad u_\vartheta = \frac{8k}{\pi r \sqrt{2\tau_2}} \left(\frac{a}{b}\right)^k \left[1 + \left(\frac{a}{b}\right)^k\right]^{-1} \sin \pi k \cos \vartheta_0 k \sin \vartheta k + O(1) \tag{4.3}$$

Analogous expressions can also be obtained for the case of a transverse incident wave.

In particular, it is clear from these expressions that the additional elastic terms have the same intensity as the acoustic terms near the front, but a different angular distribution. Thus the difference between the elastic problem and the acoustic problem is important, not only near the edge, of the wedge but also in the entire region of diffraction.

BIBLIOGRAPHY

1. Frank, Ph. and von Mises, R., Differntsial'nye i integral'nye uravneniia matematicheskoi fiziki (Differential and Integral Equations of Mathematical Physics), Chapter XII (Russian edition only), ONTI, 1937.

2. Friedlander, F.G., Sound Pulses, Cambridge Univ. Press, 1958.

3. Filippov, A.F., Difraktsiia proizvol'noi akusticheskoi volny na kline (Diffraction of an arbitrary acoustic wave by a wedge), *PMM*, Vol. 28, No. 2, 1964.

4. Filippov, A.F., Nekotorye zadachi difraktsii ploskikh uprugikh voln (Some problems of diffraction of plane elastic waves), *PMM*, Vol. 20, No. 6, 1956.

5. Sveklo, V.A. and Siukiiainen, V.A., Difraktsiia ploskoi uprugoi volny otnositel'no uglia (Diffraction of a plane elastic wave by a wedge) Doklady AN SSSR, Vol. 119, No. 6, 1958.

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